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Logicoalgebraic Foundations of Contact Mechanics

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The logicoalgebraic foundations of the Lagrangian and Hamiltonian techniques of contact mechanics are exhibited, by starting axiomatically with a classical system whose logic is a Boolean σ algebra.

1. INTRODUCTION

This paper exhibits the logicoalgebraic foundations of the Lagrangian and Hamiltonian techniques of contact mechanics.

A line of thought related in spirit to the foundational methods of quantum logic (Varadarajan, 1968 and 1970) is followed.

This line pursues a systematic transition from the physical formulation—expressed in terms of *observables* and *states* associated with a *Boolean σ algebra* (Barone and Grassini, 1983)—to the geometrical setting—expressed in terms of contact structures (Abraham and Marsden, 1978)—of classical mechanics.

Logic of a classical deterministic system with finite degrees of freedom is briefly recalled in Section 2; there we are led from a representation theorem to single out, as a privileged representation space, the *space of pure states* associated with the logic.

Kinematics of a system with *time-dependent* holonomic constraints is then developed in Section 3; there we show how the pure state space can be endowed with a classical vector bundle structure, tangent to the fibers of a configuration space-time, as soon as the geometrical description of pure states is committed to suitable *kinematical observables* (position, velocity, and absolute time).

Dynamics of a *nonconservative* system is finally studied in Section 4; there we can transfer Lagrangian and Hamiltonian contact techniques onto

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pure state space and recover a logicoalgebraic *characterization of all second-order motion equations* on configuration space-time, in terms of suitable *dynamical observables* (kinetic energy and force field).

2. LOGIC

Let $\mathscr{L}$ be a Boolean σ algebra.

$\mathscr{L}$ will be thought of as the logic of a classical physical system, whose states and observables are then defined as probability measures on $\mathscr{L}$ and σ morphisms from the Borel structure of real line R onto $\mathscr{L}$, respectively (Varadarajan, 1968).

$\mathscr{L}$ will be then assumed to be separable and atomic. Separability indeed corresponds to finiteness of degrees of freedom, i.e., to existence of finite complete systems of observables; atomicity then corresponds to determinism, i.e., to existence of the set $\mathscr{P}$ of pure states, which take strict values to observables and can be as well characterized as deterministic measures—with 0 and 1 values only—concentrated at atoms of $\mathscr{L}$ (Kronfli, 1970; Barone and Galdi, 1979).

Owing to separability, there exists a σ epimorphism

$$u: \mathscr{B}(S) \to \mathscr{L}$$

from the Borel structure of any standard Borel space S (with the power of the continuum), onto $\mathscr{L}$.

Owing to atomicity, subspace

$$P = \{x \in S / u(\{x\}) \neq 0\} \subset S$$

is a representation space of $\mathscr{L}$, i.e., a separable Borel space whose Borel structure is related to $\mathscr{L}$ by a σ isomorphism (Barone and Grassini, 1983)

$$\phi: \mathscr{B}(P) \to \mathscr{L}$$

Any other representation space P', ϕ' is Borel isomorphic with P, ϕ. In fact, if

$$\psi: P \to \mathscr{P}$$

(resp. ψ') is the bijection induced by ϕ (resp. ϕ') upon identification of atoms with pure states of $\mathscr{L}$, composition

$$\psi'^{-1} \circ \psi: P \to P'$$

acts on Borel sets according to σ isomorphism

$$\phi'^{-1} \circ \phi: \mathscr{B}(P) \to \mathscr{B}(P')$$

and therefore is a Borel isomorphism.

Consequently a unique separable Borel structure, σ isomorphic with $\mathscr{L}$, will be defined on $\mathscr{P}$ by requiring ψ (or ψ') to be a Borel isomorphism.

Then, if we call *classical logic* a separable and atomic Boolean σ algebra, we can state the following:

Lemma. Let $\mathscr{L}$ be a classical logic. Then there exists a unique—up to Borel isomorphisms—representation space of $\mathscr{L}$: it is the space $\mathscr{P}$ of pure states of $\mathscr{L}$.

3. KINEMATICS

Let $\mathscr{L}$ be a classical logic.

Kinematics is the study of preferred systems of observables, giving a geometrical description of pure states of $\mathscr{L}$ (Barone and Grassini, 1983).

To this end we consider the set of all complete systems of observables of an odd order $2n+1$, each of them being regarded as a σ epimorphism u with standard space $S = R^{2n} \times R$.

Then we define an action of the group of diffeomorphisms of R^n on the above set, through transformation law

$$u' = u \circ (\mathbf{T}k \times \mathbf{1}_R)^{-1}$$

$\mathbf{T}$ being the tangent functor, k a diffeomorphism of R^n, and $\mathbf{1}_R$ the identity map on R.

Finally we call *kinematics* on $\mathscr{L}$ an orbit K of this action, and *position*, *velocity*, and *absolute time* observables, respectively, the ones collected by any ordered system $u \in K$, for their strict values, i.e., the coordinates of the points of representation space $P \subset R^{2n} \times R$ change, after a transformation k of observables, according to the classical law $\mathbf{T}k \times \mathbf{1}_R$.

Constraints are then invariant conditions characterizing P as a subset of $R^{2n} \times R$.

Holonomic constraints are precisely expressed by

$$P = \operatorname{Ker}(\mathbf{T}_t f)$$

where f: $R^n \times R \to R^s$ is a differentiable mapping whose kernel is the Cartesian product of a submanifold $M \subset R^n$ by *time axis* R, and $\mathbf{T}_t f$ is the restriction of $\mathbf{T}f$ to unit timelike vectors tangent to $R^n \times R$.

Holonomy condition corresponds to time-dependent constraint equations on position coordinates, which single out a product manifold $M \times R \subset R^n \times R$; equations inferred from the previous ones by time derivation then identify P with the bundle of all unit timelike vectors tangent to

$M \times R$ or, through a natural isomorphism, the bundle, tangent to the fibers of $M \times R$ on R,

$$P = \mathbf{T}M \times R$$

Let K be a *holonomic kinematics.*

For any $u \in K$, bijection ψ defines a differentiable fibration π on $\mathcal{P}$, related to natural fibration τ_M of $\mathbf{T}M$ by a commutative diagram

$$\begin{array}{ccc} \mathbf{T}M \times R & \xrightarrow{\psi} & \mathcal{P} \\ {\scriptstyle \tau_M \times 1_R}\downarrow & & \downarrow{\scriptstyle \pi} \\ M \times R & \xrightarrow[\lambda \times 1_R]{} & Q \times R \end{array}$$

To require ψ to be a vector bundle isomorphism makes π a vector bundle structure on $\mathcal{P}$, isomorphic with $\mathbf{T}Q \times R$ through the commutative diagram

$$\begin{array}{ccc} \mathbf{T}Q \times R & \xrightarrow{\psi \circ (\mathbf{T}\lambda^{-1} \times \mathbf{1}_R)} & \mathcal{P} \\ & {\scriptstyle \tau_Q \times 1_R}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & Q \times R & \end{array}$$

For any other $u' \in K$, transformation law $u' = u \circ (\mathbf{T}k \times \mathbf{1}_R)^{-1}$, which leaves holonomy condition invariant, entails

$$\psi' = \psi \circ (\mathbf{T}\xi^{-1} \times \mathbf{1}_R)$$

and

$$\lambda' = \lambda \circ \xi^{-1}$$

with $\xi = k_{|M}$, and then

$$\psi' \circ (\mathbf{T}\lambda'^{-1} \times \mathbf{1}_R) = \psi \circ (\mathbf{T}\lambda^{-1} \times \mathbf{1}_R)$$

Consequently bijection ψ' still defines the same differentiable fibration π on $\mathcal{P}$, which, as a vector bundle, turns out to be canonically isomorphic with $\mathbf{T}Q \times R$—Cartesian product of *phase space* $\mathbf{T}Q$, tangent to *configuration space* Q, by time axis R.

Then, if we call *standard logic* a classical logic endowed with a holonomic kinematics, we can state the following:

Theorem. Let $\mathcal{L}$ be a standard logic. Then the space $\mathcal{P}$ of pure states of $\mathcal{L}$ is a vector bundle

$$\pi: \mathcal{P} \to Q \times R$$

tangent to the fibers of configuration space-time $Q \times R$.

4. DYNAMICS

Let $\mathscr{L}$ be a standard logic.

Dynamics is the study of *dynamical sprays* of $\mathscr{L}$, defined as vector fields X on pure state space $\mathscr{P}$ which are second-order equations on configuration space-time $Q \times R$, i.e.,

$$\mathbf{T}\pi \circ X = \mathbf{1}_{\mathscr{P}}$$

$\mathscr{P} = \mathbf{T}Q \times R$ being regarded as the subbundle of unit timelike vectors of $\mathbf{T}(Q \times R)$—the integral curves of X (at points with zero time) are just the derivatives of their π projections and these projected or base integral curves are *motions*, i.e., sections of $Q \times R$.

Contact methods first arise in the study of *inertial sprays* defined as follows.

Assume $\mathscr{L}$ to admit a *hyperregular* observable T.

If T is identified with the Borel function

$$T: \mathscr{P} \to R$$

whose range is the set of all its strict values, then hyperregularity requires T to be a differentiable function on vector bundle $\mathscr{P}$ whose fiber derivative

$$\mathbf{L}T: \mathscr{P} \to \mathscr{P}^*$$

is a (fiber-preserving) diffeomorphism onto dual bundle $\mathscr{P}^* = \mathbf{T}^*Q \times R$.

Hyperregularity allows Legendre transformation $\mathbf{L}T$ to pull natural contact form $\tilde{\omega}$ of $\mathscr{P}^*$ back to $\mathscr{P}$, there defining a contact form

$$\omega_T = \mathbf{L}T^*(\tilde{\omega})$$

Then consider the observable E_T of $\mathscr{L}$ characterized by strict values

$$E_T(p) = \mathbf{L}T(p) \cdot p - T(p) \qquad (p \in \mathscr{P})$$

and the vector field X_T on $\mathscr{P}$ uniquely determined by Hamilton's equations

$$\mathbf{i}_{X_T}\omega_T = -\mathbf{i}_{X_T}(dE_T \wedge dt), \qquad \mathbf{i}_{X_T}\, dt = 1$$

($\mathbf{i}$ denotes the interior product, $\wedge$ the exterior product, d the differentiation, and $t: \mathbf{T}Q \times R \to R$ the second projection).

X_T is an inertial spray.

The base integral curves of X_T are just the *inertial motions* defined by Hamilton's variational principle with *Lagrangian* or *kinetic energy* T (Abraham and Marsden, 1978).

Consequently, turning to the study of an arbitrary dynamical spray X, we stress the difference $X - X_T$ which causes *forced* motions.

The second order of sprays entails

$$\mathbf{T}\pi \circ (X - X_T) = 0$$

and then (as can be checked in local coordinates)

$$\mathbf{i}_{X-X_T}\omega_T = \mathbf{L}T \circ F, \qquad \mathbf{i}_{X-X_T}\, dt = 0$$

where, in particular, 1-form $\mathbf{i}_{X-X_T}\omega_T$ on $\mathcal{P}$, which verticality of $X-X_T$ reduces to a fiber bundle morphism from $\mathcal{P}$ to $\mathcal{P}^*$, has been expressed, through Legendre transformation, in terms of a *vector-valued* observable

$$F\colon \mathcal{P} \to \mathcal{P}$$

(fiber bundle endomorphism of $\mathcal{P}$) called—as in elementary point dynamics—a *force field.*

So, owing to the definition of X_T, X turns out to be the vector field on $\mathcal{P}$ uniquely determined by Lagrange's equations

$$\mathbf{i}_X\omega_T = -\mathbf{i}_{X_T}(dE_T \wedge dt) + \mathbf{L}T \circ F, \qquad \mathbf{i}_X\, dt = 1$$

Then, if we call *Lagrangian system* a standard logic endowed with a hyperregular observable, we can state the following:

Theorem. Let $\mathcal{L}$ be a Lagrangian system. Then any dynamical spray X of $\mathcal{L}$ is determined, through Lagrange's equations, by hyperregular observable T of $\mathcal{L}$ (the kinetic energy) and a unique vector-valued observable F on $\mathcal{L}$ (the force field).

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